Unitary representations of permutation groups

I. Unitary representations:

Définition: Let G be a topological group. a unitary representation of G on a Hilbert space $\mathcal{H}$ is an action $G \overset{\pi}{\curvearrowright} \mathcal{H}$ by linear isometries, ie a group morphism

$$\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$$

It is continuous if:

$$\forall \xi \in \mathcal{H}, \quad \|\pi(g)\xi - \xi\| \xrightarrow[g \to 1]{} 0 \quad (\text{sot})$$

equivalently, if:

$$\forall \xi, \eta \in \mathcal{H}, \quad \begin{array}{ccc} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & \langle \pi(g)\xi, \eta\rangle \end{array} \text{ is } C^0 \ (\text{wot})$$

Example: (left)

Let G be locally compact, μ a Haar meas. on G. Then $G \curvearrowright (G, \mu)$ gives rise to a continuous unitary representation

$$G \overset{\lambda_G}{\curvearrowright} L^2(G, \mu)$$

via $\lambda_G(g)(f) = f(g^{-1}\cdot)$ pour $g \in G$, $f \in L^2(G, \mu)$

$\rightsquigarrow$ left regular representation.

Convention: $G \overset{\pi}{\curvearrowright} \mathcal{H}$ means "π is a continuous unitary representation of G on $\mathcal{H}$".

Definition: Let $\pi : G \curvearrowright \mathcal{H}$

1) ~~Let $\pi : G \curvearrowright \mathcal{H}$~~. If $K \subseteq \mathcal{H}$ is closed subspace that is G-invariant, ie

$$\forall g \in G, \ \pi(g) K \subseteq K,$$

we say that σ, the restriction of π to $G \curvearrowright K$, is a subrepresentation of π and write $\sigma \leq \pi$.

NB: If K is stable, so is $K^\perp$:

Let $\xi \in K^\perp$ and $\eta \in K$. We have for every $g \in G$:

$$\langle \pi(g)\xi, \eta \rangle = \langle \underbrace{\xi}_{\in K^\perp}, \underbrace{\pi(g^{-1})\eta}_{\in K} \rangle = 0$$

$$\rightsquigarrow \underset{\substack{\pi \curvearrowright \\ G}}{\mathcal{H}} = \underset{\substack{\sigma \curvearrowright \\ G}}{K} \oplus^\perp \underset{\substack{\rho \curvearrowright \\ G}}{K^\perp} \qquad \pi = \sigma \oplus \rho.$$

Definition: $\pi : G \curvearrowright \mathcal{H}$ is irreducible if the only G-invariant closed subspace are the trivial ones: $\{0\}$ and $\mathcal{H}$.

$\rightsquigarrow$ Elementary blocks $\quad$ ⚠ Morphisms:

General hope: Given a topological group G:

1) • Can one decompose every representation of G into (a sum) of irreducible ones?

2) • Can one give a nice description of the list of irreducible representations of G?

3) • What does the representation theory of G ~~remembers~~ knows about G?

It turns out questions 1 and 2 are very closely related $\rightsquigarrow$ Type 1 groups and Glimm's theorem.

II. An ideal situation: compact groups

Theorem (Peter-Weyl)

Let G be a compact group.

1) Every representation of G splits a sum of irreducible subrepresentations:
$\forall \pi : G \curvearrowright \mathcal{H}$, $\exists (\mathcal{H}_i)_{i\in I}$ invariant, closed, pairwise orthogonal,
$\mathcal{H} = \overset{\perp}{\bigoplus_{i\in I}} \mathcal{H}_i$ and $\forall i$, $G \curvearrowright \mathcal{H}_i$ is irred.

2) Every irreducible is finite dimensional

3) Every irreducible is $\subseteq L^2(G, \mu)$

<u>Consequences</u>:

1) $G \longrightarrow \mathrm{Rep}(G)$ is an equivalence of categories (can rebuild G fully from the category of its representations)

$\rightsquigarrow$ Tannaka and Krein duality.

2) Every compact Lie group is a closed subgroup of $GL_n(\mathbb{C})$ for some n.

3) ~~Tout groupe compact est~~ ~~~~

Every compact group is an inverse limit of compact Lie groups

<u>Beyond compact groups</u>:

<u>Theorem</u> (Gelfand - Raikov)

Let G be locally compact ~~~~

Pour tout $g \in G \setminus \{e\}$ there exists $\pi : G \curvearrowright \mathcal{H}$ irreducible s.t $\pi(g) \neq \mathrm{Id}_{\mathcal{H}}$.

III. Permutation groups:

Definition:

Let Ω be a countable set and $Sym(\Omega)$ be the group of all permutations ~~groups~~ of Ω.

$Sym(\Omega) \subseteq \Omega^{\Omega}$ → product topology with Ω discrete.
(0-1 distance)

~~Is~~ Metrizable: $\Omega = \{\omega_n, n \in \mathbb{N}\}$

$$d(f,g) = \sum_{n \in \mathbb{N}} \frac{d(f(\omega_n), g(\omega_n))}{2^n}$$

ie $f_n \longrightarrow g \Leftrightarrow \forall \omega, \left(f_n(\omega) = g \text{ for } n \text{ large enough}\right)$

Basis of the topology:

$$O_{A,f} = \{ g \in Sym(\Omega), \forall a \in A, f(a) = g(a) \}$$

$(f \in Sym(\Omega), A \subseteq \Omega \text{ finite})$

Basis of nbhd of 1:

$$Sym_A(\Omega) = \{ g \in Sym(\Omega), \overset{\forall a \in A}{g(a) = a} \}.$$

subgroup, clopen.

Definition:

Let Ω be a ctbl set. A (closed) permutation group over Ω is a closed subgroup $G \leq Sym(\Omega)$

$\rightsquigarrow$ Induced topology, basis of nbhd at 1:

$$G_A := \{ g \in G, \forall a \in A, g(a) = a \} \quad , A \subseteq \Omega \text{ fini}$$

~~Lemme fondamental de l'anal~~

Fundamental lemma of harmonic analysis on permutation groups:

Let $G \leq Sym(\Omega)$ and let $\pi : G \curvearrowright \mathcal{H}$.

<u>Definition</u>: For $A \subseteq \Omega$ finite, set

$$\mathcal{H}_A = \{ \xi \in \mathcal{H}, \forall g \in G_A, \pi(g)\xi = \xi \}$$

<u>Lemme</u>:

1) $$\mathcal{H} = \overline{\bigcup_{A \subseteq \Omega \text{ finite}} \mathcal{H}_A}$$

2) $\forall g \in G, \forall A \subseteq \Omega$ finite,

$$\pi(g)\mathcal{H}_A = \mathcal{H}_{g(A)}$$

Preuve :

Fact : Let $K \subseteq H$ be convex, closed, $\neq \emptyset$. There exists unique $\xi \in H$ of minimal norm.

Proof :

Uniqueness : Suppose $\xi, \eta \in K$ have minimal norm M.

$$\underbrace{\left\| \frac{\xi+\eta}{2} \right\|^2}_{\in K} = \frac{\|\xi\|^2 + \|\eta\|^2 - \frac{1}{2}\|\xi - \eta\|^2}{2} < M$$

Existence : Let $(\xi_n) \in K^{\mathbb{N}}$ be such that $\|\xi_n\| \longrightarrow M$

Then :

$$\left\| \frac{\xi_n - \xi_m}{2} \right\|^2 = \frac{\|\xi_n\|^2 + \|\xi_m\|^2 - \frac{1}{2}\|\xi_n + \xi_m\|^2}{2}$$

$$= \underbrace{\frac{\|\xi_n\|^2 + \|\xi_m\|^2}{2}}_{\leq \frac{M^2 + \varepsilon + M^2 + \varepsilon}{2}} - \underbrace{\left\| \frac{\xi_n + \xi_m}{2} \right\|^2}_{\geq M^2}$$

$\leq \varepsilon$

Donc (ξ_n) de Cauchy : converge.

Back to the proof of the lemma:

1) Let $\xi \in \mathcal{H}$ and $\varepsilon > 0$.

By continuity of
$$\begin{array}{ccc} G & \longrightarrow & \mathbb{R}_+ \\ g & \longmapsto & \|\pi(g)\xi - \xi\| \end{array}$$
at 1_G,

there exists $A \subseteq \Omega$ finite such that

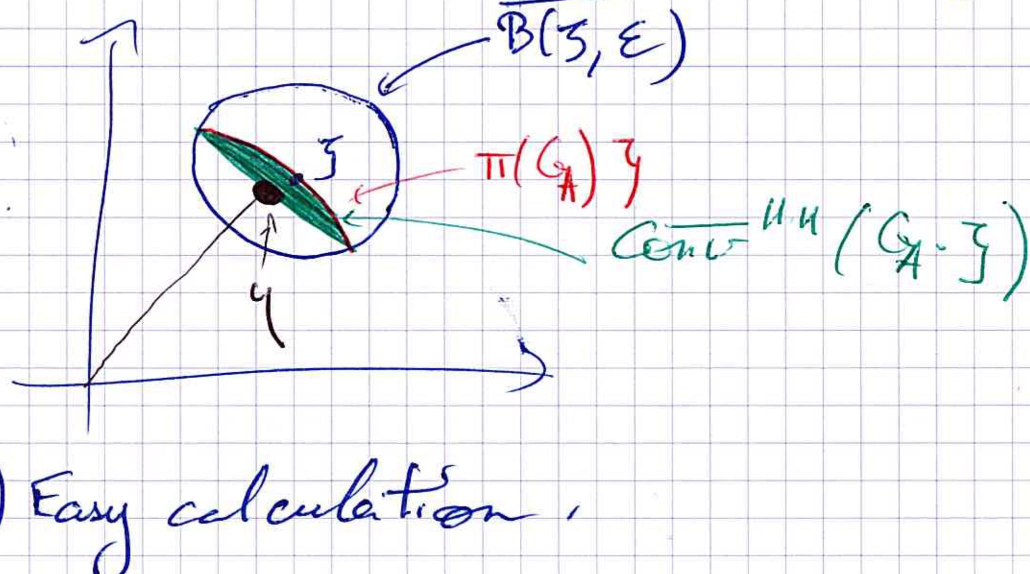
$$\forall g \in G_A, \quad \|\pi(g)\xi - \xi\| < \varepsilon$$

Apply the fact to $K := \overline{\mathrm{Conv}}^{\|\cdot\|}(G_A \cdot \xi)$: pick $\eta \in K$ of minimal norm.

Then $\forall g \in G_A$, $\begin{cases} \pi(g)\eta \in K \\ \|\pi(g)\eta\| = \|\eta\| \end{cases}$

hence $\forall g \in G_A$, $\pi(g)\eta = \eta$ ie $\eta \in \mathcal{H}_A$.

Moreover, ~~$\eta \in B$~~ $K \subseteq B(\xi, \varepsilon)$ by convexity



2) Easy calculation.